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Role of Common fixed point theorems using sequencially Contractive Mapping

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Contraction is indeed a beautiful relationship between the functions and their representations, not just an expression. Bonsall as well as Nadlerhave investigated the structure of fixed points of contraction mapping. These authors consider a sequence $\{T_n\}$ of maps defined on a metric space (X, d) into itself and study the convergence of the sequence of fixed points for uniform or pointwise convergence of $\{T_n\}$, under contraction assumptions of the maps. In this chapter we are going to introduce the concept of sequencially weak contraction using sequence of functions which are uniformly convergent to a continuous function. The concept of weak contraction is already given by Dutta et. al. In this chapter, we present proof of three main theorems. In first theorem, we prove a fixed-point theorem for sequence of functions, which generalizes the result of Beg and Abbas

The object of second and third theorem is to prove common fixed point theorem for pair of sequencially weak compatible mappings, which extends the well-known result of Moradi and

We shall require the following definitions before the statement of our theorem.

Definition 2.1 Let (X, d) is a metric space. A mapping T: X → X is called **contraction** if there exists a real number α with $0 \le \alpha < 1$ such that

(2.1) $d(Tx, Ty) \le \alpha\, d(x, y)$, for all $x, y \in X$, $x \ne y$.

Definition 2.2 A mapping $T : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if

(2.2) $d(Tx, Ty) \le d(x, y) - \varphi(d(x, y))$,

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

If we put $\varphi(t) = kt$ where $0 < k < 1$, then equation (2.2) reduces to Banach contraction principle.

Definition 2.3 A mapping $T : X \to X$, where (X, d) is a metric space, is said to be sequencially weakly contraction if

(2.3) $d(Tx, Ty) \le d(x, y) - f_n(d(x, y))$, (where $f_n : I \to R$, I is subset of R.)

where $x, y \in X$ and $f_n(t)$ is a sequence of function which converges uniformly to t, and non-decreasing function such that $f_n(t) = 0$ if and only if $t = 0$.

If we put $f_n(t) = kt$ where $0 < k < 1$ and $t = 1$, then equation (2.3) reduces to Banach contraction principle.

Definition 2.4 Two self-mappings f and g of a metric space (X, d) are said to be weakly commuting if

(2.4) $d(fgx, gfx) \le d(gx, fx)$ for all x in X.

Further, Jungck introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity.

Definition 2.5 Two self-mappings f and g of a metric space (X, d) are said to be compatible if

(2.5) $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 2.6 Let S and T be self-mapping of a non-empty set X . The mapping T and S are weakly compatible if

(2.6) $TSx = STx$ whenever $Tx = Sx$.

Theorem 2.7 Let T be a self mapping on a complete metric space (X, d) satisfying (2.3) i.e., $d(Tx, Ty) \le d(x, y) - f_n(d(x, y))$, (where $f_n: I \to R$, I is subset of R.)

where $x, y \in X$ and $f_n(t)$ is a sequence of function which converges uniformly to t, and non-decreasing function such that $f_n(t) = 0$ if and only if $t = 0$, then T has a unique fixed point.

Proof. From equation (2.3), we have

$d(Tx, Ty) \leq d(x, y) - f_n(d(x, y)) = (I - f_n)\, d(x, y)$

Since $f_n(t)$ is a sequence of function which converges uniformly to t and hence $(I - f_n)(t)$ uniformly converges to 0 .
Using Banach Contraction Principle, we have a unique fixed point.

Theorem 2.8 Let (X, d) be a complete metric space and T be self-maps of X satisfying (2.3) and the following :

(2.7) T is continuous and

(2.8) $T(x_{n-1}) = x_n$, $n \geq 1$

Then T has a unique fixed point $x \in X$.

Proof. Consider the sequence $\{x_n\}$given by $x_n = T(x_{n-1})$, $n \geq 1$
First we prove that $\{x_n\}$is a cauchy's sequence in X.
for $m < n$, we use the triangle inequality and note that,

(2.9) $d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) \ldots\ldots\ldots + d(x_{n-1}, x_n)$

Using equations (2.8) and (2.3), we have

$$d(x_p, x_{p+1}) = d(T(x_{p-1}), T(x_p))$$
$$\leq d(x_{p-1}, x_p) - f_n(d(x_{p-1}, x_p))$$
$$= ((I - f_n))\, d(x_{p-1}, x_p)$$

for any positive integer $p \geq 1$,using the inequality repeatedly ,we obtain

$$d(x_p,x_{p+1}) \leq ((I - f_n))d(x_{p-1},x_p)$$
$$\leq ((I - f_n))^2 d(x_{p-2},x_{p-1})$$
$$\leq ((I - f_n))^p\, d(x_0,x_1)$$

Hence,

(2.10) $d(x_p, x_{p+1}) \leq ((I - f_n))^p\, d(x_0,x_1)$

Using equation (2.9) in (2.10), we have

$d(x_m,x_n) \leq d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) + d(x_{m+2},x_{m+3}) + \ldots\ldots\ldots\ldots + d(x_{n-1},x_n)$

$\leq ((I - f_n))^m\, d(x_0,x_1) + ((I - f_n))^{m+1}\, d(x_0,x_1) + \ldots\ldots\ldots + ((I - f_n))^{n-1}\, d(x_0,x_1)$

$= (((I - f_n))^m + ((I - f_n))^{m+1} + \cdots\cdots + ((I - f_n))^{n-1})\, d(x_0,x_1))$

$\leq (((I - f_n))^m + ((I - f_n))^{m+1} + \cdots\cdots + ((I - f_n))^{n-1} + \ldots\ldots)\, d(x_0,x_1)$

$= ((I - f_n))^m \;(f_n) d(x_0,x_1)$

Since $f_n(t)$ is a sequence of function which converges uniformly to t and hence $(I - f_n)(t)$ uniformly converges to 0 .
Hence, $\{x_n\}$ is a cauchy sequence in metric space.
Since, given metric space is complete, this sequence has a limit, say x which belongs to metric space. It follows that

$$x = \lim_{n\to\infty} T(x_{n-1})$$
$$= T(\lim_{n\to\infty} x_{n-1}) \qquad \text{[T is continuous]}$$
$$= T(x)$$

And thus, x is a fixed point of T.

Uniqueness:
Let x and z both are fixed points of T, we have
$d(x, z) = d(Tx, Tz) \leq (I - f_n)\, d(x, z)$ [using equation (2.3)]
i.e., $d(x,z) \rightarrow 0$, as $f_n(t)$ is a sequence of function which converges uniformly to t.
we must have, $x = z$.

Theorem 2.9 Let (X, d) be a metric space and let $T : X \rightarrow X$ be a mapping such that,

(2.11) $d(T^m x, T^m y) \leq d(x, y) - f_n(d(x, y))\ \forall\, x, y \in X$

for some $m \geq 1$, where $f_n(t)$ is a sequence of function which converges uniformly to t and non-decreasing function such that $f_n(t) = 0$ if and only if $t = 0$, then T has a unique fixed point.

Proof. Using theorem 2.8, T^m has a unique fixed point.
Thus, $z = T^m(z)$
Implies that,

(2.12) $T(z) = T(T^m(z)) = T^m(T(z))$

Thus T(z) is a fixed point of T^m .
Hence, by uniqueness of such fixed points $z = T(z)$.
And thus, z is a fixed point of T.

Example 2.10 Consider the space, $X = \{x \in \mathbb{R} \mid x \geq 1\}$ with metric,
$d(x, y) = |x - y|\ \forall\, x, y \in X$ and $T : X \rightarrow X$ is given by $T(x) = x + \frac{1}{x}$.
Then an easy computation shows that

$$d(Tx, Ty) = \frac{xy-1}{xy}|x-y| = |x-y| - \frac{1}{xy}|x-y|$$

$$\left\{\begin{array}{c} \because T(x) = \dfrac{x^2+1}{x},\ T(y) = \dfrac{y^2+1}{y} \\ d(Tx,Ty) = \left|\dfrac{x^2+1}{x} - \dfrac{y^2+1}{y}\right| = \left|\dfrac{x^2y+y-xy^2-x}{xy}\right| \\ \left|\dfrac{xy(x-y)-(x-y)}{xy}\right| \\ \left[\dfrac{xy-1}{xy}|x-y|\right] \end{array}\right\}$$

$d(Tx, Ty) < |x - y| = d(x, y)$

on other hand there $\nexists\, f_n(t)$ is a sequence of function which converges uniformly to t such that,
$d(T(x), T(y)) \leq d(x, y) - f_n(d(x, y))\ \forall\, x, y \in X$
and one may verify that T has no fixed point in X.
The following theorem is the generalized result of Beg et. al. [39] using sequence of function as:

Theorem 2.11 Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality

(2.13) $\psi(d(Tx, Ty)) \le \psi(d(x, y)) - f_n(d(x, y))$

(f_n:I (interval or subset of R) $\to$ R) where $f_n(t)$ is a monotonically non-decreasing sequence of function which converges uniformly to $\psi(t)$.

Where $\psi : [0,\infty) \to [0,\infty)$ is continuous and monotonically non-decreasing and continuous function. Then T has a unique fixed point.

Proof. For any $x_0 \in X$,we construct a sequence $\{x_n\}$ by,

$x_n = Tx_{n-1}$, n = 1,2,3,4…….

substituting $x = x_{n-1}$ and $y = x_n$ in equation (2.13), we obtain

(2.14) $\psi(d(x_n, x_{n+1})) \le \psi(d(x_{n-1}, x_n)) - f_n(d(x_{n-1}, x_n))$

Which implies,

(2.15) $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)$ (using monotonic property of ψ -function)

it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing and consequently there exist $r \ge 0$ such that

(2.16) $d(x_n, x_{n+1}) \to r$ as $n \to \infty$

letting $n \to \infty$ in equation (2.14), we obtain

(2.17) $\psi(r) \le \psi(r) - \psi(r)$

since, $\lim_{n\to\infty} f_n(r) = \psi(r)$

Which is a contradiction unless r = 0 ,since $\psi(r) \ge 0$. Hence

(2.18) $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$

we next prove that $\{x_n\}$ is a cauchy sequence.

If possible let $\{x_n\}$ is not cauchy sequence then there exist $\varepsilon > 0$ for which we can find subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

(2.19) $d(x_{m(k)},x_{n(k)}) \ge \varepsilon$

further corresponding to m(k), we can choose n(k) in such a way that it is a smallest integer with $n(k) > m(k)$ and satisfying equation (2.19), then

(2.20) $d(x_{m(k)},x_{n(k)-1}) < \varepsilon$

then we have,

(2.21) $\varepsilon \le d(x_{m(k)},x_{n(k)}) \le d(x_{m(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)}) < \varepsilon + d(x_{n(k)-1},x_{n(k)})$

letting $k \to \infty$ and using equation (2.18), we have

(2.22) $\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) < \varepsilon$

again,

(2.23) $d(x_{n(k)-1},x_{m(k)-1}) \le d(x_{n(k)-1},x_{n(k)}) + d(x_{n(k)},x_{m(k)}) + d(x_{m(k)},x_{m(k)-1})$

letting $k \to \infty$ in the above inequalities and using equations (2.18) and (2.22), we get

(2.24) $\lim_{k\to\infty} d(x_{n(k)-1},x_{m(k)-1}) = \varepsilon$

setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in equation (2.13) and using equation (2.19), we obtain

(2.25) $\psi(\varepsilon) \le \psi(d(x_{m(k)}, x_{n(k)}))$ [since $Tx_{n-1} = x_n$]

$\le \psi(d(x_{m(k)-1}, x_{n(k)-1})) - f_n(d(x_{m(k)-1}, x_{n(k)-1}))$

letting $k \to \infty$ in the above inequalities and using equations (2.22) and (2.24), we obtain

(2.26) $\psi(\varepsilon) < \psi(\varepsilon) - f_n(\varepsilon)$

Which is a contradiction if $\varepsilon > 0$.

Since $f_n(t)$ converges uniformly to $\psi(\varepsilon)$.

This shows that $\{x_n\}$ is a cauchy sequence and hence is convergent in the complete metric space X.

(2.27) Let $x_n \to z$ (say) as $n \to \infty$

Substituting $x = x_{n-1}$ and $y = z$ in equation (2.13), we obtain

(2.28) $\psi(d(x_n,Tz)) \le \psi(d(x_{n-1},z)) - f_n(d(x_{n-1},z))$ [since $Tx_{n-1} = x_n$]

letting $n \to \infty$, using equation (2.27) and continuity of ψ and continuity of f_n at infinity we have

$\psi(d(z,Tz)) \le \psi(0) - \lim_{n\to\infty} \{f_n(0)\}$

$\le \psi(0) - \psi(0)$

$= 0$

Which implies, $\psi(d(z, Tz)) = 0$

(2.29) i.e., $d(z,Tz) = 0$

or

(2.30) $z = Tz$

To prove uniqueness of fixed point, let z_1 and z_2 are two fixed points of T.

Putting $x = z_1$ and $y = z_2$ in equation (2.13), we have

$\psi(d(Tz_1, Tz_2)) \le \psi(d(z_1, z_2)) - f_n(d(z_1, z_2))$

or $\psi(d(z_1, z_2)) \le \psi(d(z_1, z_2)) - f_n(d(z_1, z_2))$ [using equation (2.30)]

(2.31) $\psi(d(z_1, z_2)) \le 0$

Since $f_n(t)$ converges uniformly to $\psi(t)$. Hence

(2.32) $d(z_1, z_2) = 0$, i.e., $z_1 = z_2$.

this proves the uniqueness of fixed point.

The following result is the generalized result of Moradi et. al. [89] on metric space for pair of sequencially weak compatible mappings.

Theorem 2.12 Let f and g be self mappings on a metric space (X, d) satisfying the followings:

(2.33) $gX \subset fX$,

(2.34) gX or fX is complete,

(2.35) $\psi(d(gx,gy)) \le \psi(d(fx, fy)) - f_n(d(fx,fy))$

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(f_n : I (interval or subset of R) $\rightarrow$ R) for all x, y $\in$ X

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is mappings with $\psi(0) = 0$, $f_n(t) > 0$ also, $f_n(t)$ is a uniformally convergent sequence which converges to ψ (t) and $\psi(t) > 0$ for all t > 0.

Suppose also that either

(a) ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} f_n(t_n) = 0$ or

(b) ψ is monotonic non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$.

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From equation (2.33), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by

$y_n = fx_{n+1} = gx_n$, n = 0, 1, 2, ...

Moreover, we assume that if $y_n = y_{n+1}$ for some n $\in \mathbb{N}$, then there is nothing to prove.

Now, we assume that $y_n \neq y_{n+1}$ for all n $\in \mathbb{N}$. Substituting $x = x_{n+1}$ and $y = x_n$ in equation (2.35), we have

(2.36) $\psi(d(y_{n+1}, y_n)) = \psi(d(gx_{n+1}, gx_n))$

$\leq \psi(d(fx_{n+1}, fx_n)) - f_n(d(fx_{n+1}, fx_n))$

$= \psi(d(y_n, y_{n-1})) - f_n(d(y_n, y_{n-1}))$ [using $y_n = fx_{n+1}$]

for all n $\in \mathbb{N}$ and hence, the sequence $\{\psi(d(y_{n+1}, y_n))\}$ is monotonic decreasing and bounded below.

Thus, there exists r $\geq$ 0 such that $\lim_{n\to\infty} \psi(d(y_{n+1}, y_n)) = r$.

From equation (2.36), we deduce that

(2.37) $0 \leq f_n(d(y_n, y_{n-1}))$

$\leq \psi(d(y_n, y_{n-1})) - \psi(d(y_{n+1}, y_n))$.

Letting $n \rightarrow \infty$ in the above inequality, we get $\lim_{n\to\infty} f_n(d(y_n, y_{n-1})) = 0$.

If (a) holds, then by hypothesis $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

If (b) holds, then from equation (2.37), we have $d(y_{n+1}, y_n) < d(y_n, y_{n-1})$, for all n$\in \mathbb{N}$.

Hence $\{d(y_{n+1}, y_n)\}$ is monotonically decreasing and bounded below sequence.

By hypothesis, $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

Therefore, in every case, we conclude that

(2.38) $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

Now, we claim that $\{y_n\}$ is a cauchy sequence. Indeed, if it is false, then there exists $\varepsilon > 0$ and the subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that n(k) is minimal in the sense that n(k) > m(k) > k and $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$ and by using the triangular inequality, we obtain

$\varepsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$

$\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$

(2.39) $< 2d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)})$ [since n(k) is minimal]

Letting $k \rightarrow \infty$ in the above inequality and using equation (2.38), we get

(2.40) $\lim_{k\to\infty} (d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} (d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon$.

For all k$\in \mathbb{N}$, from equation (2.35), we have

(2.41) $\psi(d(y_{m(k)}, y_{n(k)})) \leq \psi(d(y_{m(k)-1}, y_{n(k)-1})) - f_n(d(y_{m(k)-1}, y_{n(k)-1}))$

If (a) holds, then

$\lim_{k\to\infty} \psi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \psi(\varepsilon)$,

Now, from equation (2.41), we conclude that

$\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1}) = 0$.

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction. [using equation (2.40)]

If (b) holds, then from equation (2.41), we have $\varepsilon < d(y_{m(k)}, y_{n(k)}) < d(y_{m(k)-1}, y_{n(k)-1})$ and so $d(y_{m(k)}, y_{n(k)}) \rightarrow \varepsilon^+$ and $d(y_{m(k)-1}, y_{n(k)-1}) \rightarrow \varepsilon^+$ as $k \rightarrow \infty$.

Hence $\lim_{k\to\infty} \psi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \psi(\varepsilon^+)$,

where $\psi(\varepsilon^+)$ is the right limit of ψ at ε.

Therefore, from equation (2.41), we get

$\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1})) = 0$.

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction.

Thus $\{y_n\}$ is a cauchy sequence.

Since fX is complete, so there exists a point z $\in$ fX such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = z$.

Now, we show that z is the common fixed point of f and g. Since z $\in$ fX, so there exists a point p $\in$ X such that fp = z.

If (a) holds, then from equation (2.35), for all n $\in \mathbb{N}$, we have

$\psi(d(fp, gp)) = \lim_{n\to\infty} \psi(d(gp, gx_n))$

$\leq \lim_{n\to\infty} \psi(d(fp, fx_n)) - \lim_{n\to\infty} f_n(d(fp, fx_n))$

$\leq \lim_{n\to\infty} \psi(d(fp, fx_n))$.

(2.42) $\psi(d(fp, gp)) \leq \lim_{n\to\infty} \psi(d(fp, fx_n))$.

Using condition (a) and $\lim_{n\to\infty} y_n = z$, we get

ψ(d(fp, gp)) $\leq$ ψ(d(z, z)) = ψ(0) = 0 and so d(gp, fp) = 0 (note that f_n and ψ are non-negative with $f_n(0) = \psi(0) = 0$), which implies that gp = fp = z.
If (b) holds, then from equation (2.35), we have
ψ(d(fp, gp)) = $\lim_{n\to\infty} \psi$(d(gp,gx_n)) $\leq$ $\lim_{n\to\infty} \psi$(d(fp, fx_n)) - $\lim_{n\to\infty}$ f_n(d(fp,fx_n))
(2.43) ψ(d(fp, gp)) =0 (since f_n converges uniformly to ψ)
d(fp, gp) = 0, which implies that fp = gp = z(say).
Now, we show that z = fp = gp is a common fixed point of f and g. Since fp = gp and f, g are weakly compatible maps, we have fz = fgp = gfp = gz.
We claim that fz = gz = z.
Let, if possible, gz $\neq$ z.
If (a) holds, then from equation (2.35), we have
ψ(d(gz, z)) = ψ(d(gz, gp))
$\leq \psi$(d(fz, fp)) – f_n(d(fz,fp))
= ψ(d(gz, z)) – f_n(d(gz,z))
< ψ(d(gz, z)), a contradiction.
If (b) holds, then we have
d(gz, z) < d(gz, z), a contradiction. Hence
(2.44) gz = z = fz
So z is the common fixed point of f and g.
For the uniqueness, let u be another common fixed point of f and g, so that fu = gu = u.
We claim that z = u. Let, if possible, z $\neq$ u.
If (a) holds and n$\to\infty$ then from equation (2.35), we have
ψ(d(z, u)) = ψ(d(gz, gu))
$\leq \psi$(d(fz, fu)) – f_n(d(fz,fu))
= ψ(d(z, u)) – f_n(d(z,u))
< ψ(d(z, u)), a contradiction.
If (b) holds, then we have
d(z, u) < d(z, u), a contradiction.
Thus, d(z, u) = 0 i.e., we get z = u.
Hence z is the unique common fixed point of f and g.
Now, we prove our result on metric space for pair of sequencially weak compatible mappings which generalized the result of Moradi et. al.

Theorem 2.13 Let f and g be self mappings of a metric space (X, d) satisfying equations (2.33), (2.34) and the following:
(2.45) ψ(d(gx, gy)) $\leq \psi$(N(fx, fy)) – f_n(N(fx, fy)),(sequentially weak contractive mapping)
where N(fx, fy) = max{d(fx, fy), d(fx, gx), d(fy, gy), $\frac{d(fx,gy)+d(fy,gx)}{2}$},
for all x,y $\in$ X, where f_n(t) is a monotonically non-decreasing sequence of function which converges uniformly to $\psi(t)$ and $f_n(0) = 0$ and $f_n(t) > 0$ for all t > 0 and $\lim_{n\to\infty} t_n = 0$, if {$t_n$} is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$ and ψ: [0, ∞) → [0, ∞) is a mapping with $\psi(0) = 0$ and ψ (t) > 0 for all t > 0.
Suppose also that either
(c) ψ is continuous
or
(d) ψ is monotone non-decreasing and for all k > 0, $f_n(k) > \psi(k) - \psi(k^-)$, where $\psi(k^-)$ is the left limit of ψ at k.
Then f and g have a unique point of coincidence in X. Moreover, if f and g are sequentially weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in$ X. From equation (2.33), we can construct sequences {x_n} and {y_n} in X by
$y_n = fx_{n+1} = gx_n$, n = 0, 1, 2, ...
Moreover, we assume that if $y_n = y_{n+1}$ for some n $\in \mathbb{N}$, then there is nothing to prove.
Now, we assume that $y_n \neq y_{n+1}$ for all n $\in \mathbb{N}$.
Putting x = x_{n+1} and y = x_n in equation (2.45), we have
(2.46) ψ(d(y_{n+1}, y_n)) $\leq \psi$(N(y_n, y_{n-1})) – f_n((N(y_n, y_{n-1})),
where
(2.47) N(y_n, y_{n-1}) = max{ d(y_n, y_{n-1}), d(y_n, y_{n+1}), d(y_{n-1}, y_n), $\frac{d(y_n,y_n)+d(y_{n-1},y_{n+1})}{2}$}.
If d(y_n, y_{n-1}) < d(y_n, y_{n+1}), then from equation (2.47) and $y_n \neq y_{n+1}$, we conclude that
(2.48) ψ(d(y_{n+1}, y_n)) $\leq \psi$(d(y_n, y_{n+1})) – f_n(d(y_n, y_{n+1})) < ψ(d(y_n, y_{n+1})),
a contradiction.
Therefore, d(y_n, y_{n+1}) $\leq$ d(y_n, y_{n-1}).
Hence, the sequence {(d(y_n, y_{n+1}))} is monotonically decreasing and bounded below.
From equations (2.46) and (2.47), we have
(2.49) ψ(d(y_{n+1}, y_n)) $\leq \psi$(d(y_n, y_{n-1})) – f_n(d(y_n, y_{n-1})).
Therefore, the sequence {ψ(d(y_{n+1}, y_n)) } is monotonically decreasing and bounded below. Thus, there exists r $\geq$ 0 such that $\lim_{n\to\infty} \psi$(d(y_{n+1}, y_n)) = r.
From equation (2.49), we have
(2.50) $\lim_{n\to\infty} f_n$((d(y_n, y_{n-1})) = 0, implies that, $\lim_{n\to\infty}$ d(y_{n+1}, y_n) = 0.
Now, we claim that {y_n} is a cauchy sequence. Indeed, if it is false, then there exists ε > 0 and the subsequences {$y_{m(k)}$} and {$y_{n(k)}$} of {y_n} such that n(k) is minimal in the sense that n(k) > m(k) > k

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and $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$ and by using the triangular inequality, we obtain

$\varepsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$

$\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$

(2.51) $< 2\, d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)})$. [since n(k) is minimal]

Letting $k \to \infty$ in the above inequality and using equation (2.51), we get

(2.52) $\lim_{k\to\infty} (d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} (d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon$.

From equation (2.45), for all $k \in \mathbb{N}$, we have

(2.53) $\psi(d(y_{m(k)}, y_{n(k)})) \leq \psi(N(y_{m(k)-1}, y_{n(k)-1})) - f_n(N(y_{m(k)-1}, y_{n(k)-1}))$,

where

(2.54) $N(y_{m(k)-1}, y_{n(k)-1}) = \max\{ d(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)}), d(y_{n(k)-1}, y_{n(k)}), \frac{d(y_{m(k)-1}, y_{n(k)}) + d(y_{n(k)-1}, y_{m(k)})}{2}\}$

If equations (2.52) and (2.54) holds, then we conclude that $\lim_{k\to\infty} (N(y_{m(k)-1}, y_{n(k)-1})) = \varepsilon$.

If (c) holds, i.e., ψ is continuous, then

$\lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \lim_{k\to\infty} \psi(N(y_{m(k)-1}, y_{n(k)-1})) = \psi(\varepsilon)$,

and hence from equation (2.53), we conclude that $\lim_{k\to\infty} f_n(N(y_{m(k)-1}, y_{n(k)-1})) = 0$.

Since $N(y_{m(k)-1}, y_{n(k)-1})$ is bounded, we conclude that $\lim_{k\to\infty} N(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction.

If (d) holds, i.e., ψ is monotone non-decreasing, then from equation (2.53), we have

$\varepsilon < d(y_{m(k)}, y_{n(k)}) < N(y_{m(k)-1}, y_{n(k)-1})$ for all $k \in \mathbb{N}$, and so

$d(y_{m(k)}, y_{n(k)}) \to \varepsilon^+$ and $N(y_{m(k)-1}, y_{n(k)-1}) \to \varepsilon^+$ as $k \to \infty$.

Hence, $\lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \lim_{k\to\infty} \psi(N(y_{m(k)-1}, y_{n(k)-1})) = \psi(\varepsilon^+)$, where $\psi(\varepsilon^+)$ is the right limit of ψ at ε.

Therefore, from equation (2.53), we have

$\lim_{k\to\infty} f_n(N(y_{m(k)-1}, y_{n(k)-1})) = 0$.

Since $\{N(y_{m(k)-1}, y_{n(k)-1})\}$ is bounded, we conclude that $\lim_{k\to\infty} N(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction.

Thus $\{y_n\}$ is a cauchy sequence.

Since fX is complete, so there exists a point $z \in fX$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = z$.

Now, we show that z is the common fixed point of f and g.

Since $z \in fX$, so there exists a point $p \in X$ such that $fp = z$.

We claim that $fp = gp$. Let, if possible, $fp \neq gp$.

From equation (2.45), we have

$\psi(d(gp, fp)) = \lim_{n\to\infty} \psi(d(gp, gx_n))$

$\leq \lim_{n\to\infty} \psi(N(fp, fx_n)) - \lim_{n\to\infty} f_n((N(fp, fx_n))$,

$= \psi(d(fp, gp)) - f_n((d(fp, gp))$,

since $N(fp, fx_n) = \max\{d(fp, fx_n), d(fp, gp), d(fx_n, gx_n), \frac{d(fp,gx_n)+ d(fx_n,gp)}{2}\}$.

Letting limit as $n \to \infty$, we have

$\lim_{n\to\infty} N(fp, fx_n) = \max\{d(fp, z), d(fp, gp), d(z, z), \frac{d(fp,z)+ d(z,gp)}{2}\}$

$= \max\{0, d(fp, gp), 0, \frac{d(fp,gp)}{2}\} = d(fp, gp)$.

If (c) holds, then we have

$\psi(d(gp, fp)) < \psi(d(gp, fp))$, a contradiction.

If (d) holds, then we have

$d(gp, fp) < d(gp, fp)$, a contradiction.

Hence, $fp = gp = z$.

Now we show that $z = fp = gp$ is a common fixed point of f and g. Since $fp = gp$ and f, g are weakly compatible maps, we have $fz = fgp = gfp = gz$.

We claim that $fz = gz = z$.

Let, if possible $gz \neq z$

From equation (2.45), we have

$\psi(d(gz, z)) = \psi(d(gz, gp))$

$\leq \psi(N(fz, fp)) - f_n((N(fz, fp))$

$= \psi(d(gz, z)) - f_n((d(gz, z))$,

Since $N(fz, fp) = \max\{d(fz, fp), d(fz, gz), d(fp, gp), \frac{d(fz,gp)+ d(fp,gz)}{2}\}$

$= \max\{d(gz, z), d(gz, gz), d(gp, gp), \frac{d(gz,z)+ d(z,gz)}{2}\}$

$= d(gz, z)$.

If (c) holds, then we have

$\psi(d(gz, z)) < \psi(d(gz, z))$, a contradiction.

If (d) holds, then we have

$d(gz, z) < d(gz, z)$, a contradiction.

Hence $gz = z = fz$, so z is the common fixed point of f and g.

For the uniqueness, let u be another common fixed point of f and g, so that $fu = gu = u$.

We claim that $z = u$.

Let, if possible, $z \neq u$.

From equation (2.45), we have

$\psi(d(z, u)) = \psi(d(gz, gu))$

$\leq \psi(N(fz, fu)) - f_n((N(fz, fu))$

$= \psi(d(fz, fu)) - f_n((d(fz, fu))$, since $N(fz, fu) = d(fz, fu)$.

$= \psi(d(z, u)) - f_n(d(z, u))$.

If (c) holds, then we have

$\psi(d(z, u)) < \psi(d(z, u))$, a contradiction.

If (d) holds, then we have

$d(z, u) < d(z, u)$, a contradiction.

Thus, we get $z = u$.

Hence z is the unique common fixed point of f and g.

Example 2.14 Let $X = [0, 1]$ be Euclidean metric space with $d(x, y) = |x - y|$ for all x, y in X and let $gx = (\frac{1}{5})x$ and $fx = (\frac{3}{5})x$ for each $x \in X$. Then $d(gx, gy) = (\frac{1}{5})|x - y|$ and $d(fx, fy) = (\frac{3}{5})|x - y|$.

Let $\psi(t) = 5t$ and $f_n(t) = 25nt/(5n+t)$. Then

$\psi(d(gx, gy)) = \psi((\frac{1}{5})|x - y|) = |x - y|$

$\psi(d(fx, fy)) = \psi((\frac{3}{5})(|x - y| = 5(\frac{3}{5})|x - y| = 3|x - y|$

$f_n((d(fx, fy)) = 15n|x - y| /(5n+|x - y|)$.

Also $f_n(x)$ is a sequence of function which uniformly converges to $\psi(x)$.

Now

$\psi(d(fx, fy)) - f_n((d(fx, fy)) = 3|x - y|-15n|x - y| /(5n+|x - y|)$

$= 3|x - y|[1-5n /(5n+|x - y|)]$

And $[1-5n /(5n+|x - y|)] \geq 0$, if n approaches to infinity.

So $\psi(d(gx, gy)) < \psi(d(fx, fy)) - f_n((d(fx, fy))$.

From here, we conclude that f, g satisfy the relation equation (2.35).

Also $gX = [0, \frac{1}{5}] \subseteq [0, \frac{3}{5}] = fX$, f and g are weakly compatible and 0 is the unique common fixed point of f and g.

Conclusion

In this chapter some important theorems have been successfully proved, taking sequence of function which is uniformly convergent to a continuous function. In the above consideration, using the sequentially convergent mapping we have proved many generalization of already well known theorems related to fixed point in a complete metric space.

It is surprising to see that some other generalization hold true for the already existing theorems on common fixed in a complete metric space and correspondingly many other generalization can be proved. An example has been quoted to prove the desired result.

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